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Jacobi Polynomial Expansions

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The differential operator generated by the Jacobi differential equation

$$(1-x^2)\,y'' + [\beta-\alpha-(\alpha+\beta+2)\,x]\,y' + n(\alpha+\beta+n+1)\,y=0, \qquad x\in[-1,1]$$

is considered for *all* α and β in both the right and left definite spaces. Shifted Jacobi operators when $\alpha<1$, $\beta>-1$, when $\alpha>-1$, $\beta<1$, and when $\alpha<1$, $\beta<1$, and the classical Jacobi operator with $\alpha>-1$, $\beta>-1$ are introduced. We show that all Jacobi operators are self-adjoint in both spaces. The spectral resolutions of shifted Jacobi differential operators are given by comparing them to the classical Jacobi polynomial expansion.

1. INTRODUCTION

This paper is devoted to eigenfunction expansions associated with the Jacobi Polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$. The classic expansion, associated with the Jacobi operator

$$L_g=[-((1-x)^{\alpha+1}(1+x)^{\beta+1}y')'+(1-x)^{\alpha}(1+x)^{\beta}y]/(1-x)^{\alpha}(1+x)^{\beta},$$

whose domain is determined in part by singular boundary conditions

$$\lim_{x\to 1}(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)=0$$

$$\lim_{x\to -1}(1-x)^{\alpha}(1+x)^{\beta+1}y'(x)=0,$$

is well known [12] when $\alpha>-1$, $\beta>-1$, when set in the classic Hilbert space $L^2(-1,1:(1-x)^{\alpha}(1+x)^{\beta})$, generated by the inner product

$$\langle y,z\rangle_{L^2}=\int_{-1}^{1}y(x)\,\overline{z(x)}(1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

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Left unanswered is what happens when α and/or $\beta \leqslant -1$. We show that even under these circumstances there are eigenfunction expansions associated with the Jacobi operator and the Jacobi polynomials, but in a different way. Interestingly this problem seems to have stood for a long, long time.

There is an additional setting in which the various Jacobi expansions may be set. The setting in L^2, cited above, may be called right-definite since it is generated by the weight function lying on the right side of the homogeneous Jacobi differential equation

$$\begin{aligned} &-((1-x)^{\alpha+1}(1+x)^{\beta+1}y')' + (1-x)^{\alpha}(1+x)^{\beta}y \\ &\quad = (n^2+\alpha n+\beta n+n+1)(1-x)^{\alpha}(1+x)^{\beta}y. \end{aligned}$$

If this is multiplied by $\bar{z}(x)$ and integrated from -1 to 1, we find

$$\begin{aligned} &\int_{-1}^{1} [-((1-x)^{\alpha+1}(1+x)^{\beta+1}y')' + (1-x)^{\alpha}(1+x)^{\beta}y]\,\bar{z}\,dx \\ &\quad = (n^2+\alpha n+\beta n+1)\int_{-1}^{1} y\bar{z}(1-x)^{\alpha}(1+x)^{\beta}\,dx. \end{aligned}$$

The L^2 inner product lies on the right side.

The left side also generates an inner product. If the first term is integrated by parts, and the boundary terms vanish, it becomes

$$\langle y, z\rangle_{H^1} = \int_{1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1}y'\bar{z}' + (1-x)^{\alpha}(1+x)^{\beta}y\bar{z}]\,dx.$$

This generates a Sobolev space which we denote by $H^1(-1,1;(1-x)^{\alpha}(1+x)^{\beta+1};(1-x)^{\alpha}(1+x)^{\beta})$.

We show that in this new setting the Jacobi operator is also self-adjoint, and that the right-definite eigenfunction expansions involving Jacobi polynomials remain valid for all α and β in a left-definite setting.

We note that not only does the classic orthogonality condition

$$\begin{aligned} &\int_{1}^{1} P_n^{(\alpha,\beta)}(x)\,P_m^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx \\ &\quad = 2^{1+\alpha+\beta}\frac{\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)\,\delta_{mn}}{n!\,(1+\alpha+\beta+2n)\,\Gamma(1+\alpha+\beta+n)}, \end{aligned}$$

where

$$\delta_{mn} = \begin{cases} 0, & m \neq n, \\ 1, & m = n, \end{cases}$$

hold, but, in addition,

$$\int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1} P_n^{(\alpha,\beta)}(x)' P_m^{(\alpha,\beta)}(x)'$$

$$+(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)]\,dx$$

$$=\frac{(n^2+\alpha n+\beta n)\,2^{1+\alpha+\beta+n}\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)\,\delta_{mn}}{n!\,(1+\alpha+\beta+2n)\,\Gamma(1+\alpha+\beta+n)}.$$

This, left-definite orthogonality, is less well known.

In the right-definite case, the spectral resolutions of the Jacobi differential operator for $\alpha<1$, $\beta>-1$; $\alpha>-1$, $\beta<1$; and $\alpha<1$, $\beta<1$ are evaluated by comparing them to the classical case $\alpha>-1$, $\beta>-1$. It is shown that

(1) The cases $\alpha>-1$, $\beta>-1$ and $\alpha<1$, $\beta>-1$ are equivalent under the transformation $y(x)=(1-x)^{-\alpha}z(x)$.

(2) The cases $\alpha>-1$, $\beta>-1$ and $\alpha>-1$, $\beta<1$ are equivalent under the transformation $y(x)=(1+x)^{-\beta}z(x)$.

(3) The cases $\alpha>-1$, $\beta>-1$ and $\alpha<1$, $\beta<1$ are equivalent under the transformation $y(x)=(1-x)^{-\alpha}(1+x)^{-\beta}z(x)$.

This allows us to compare the α-shifted Jacobi equation with $\alpha<1$, $\beta>-1$, the β-shifted Jacobi equation with $\alpha>-1$, $\beta<1$, the $\alpha\beta$-shifted Jacobi equation with $\alpha<1$, $\beta<1$, with the classical Jacobi equation with $\alpha>-1$, $\beta>-1$. We present the necessary conditions for the Jacobi operator to be self-adjoint in all four cases. Then we show

(1) If $G(x)\in L^2(-1,1;(1-x)^{\alpha}(1+x)^{\beta})$ with $\alpha<1$, $\beta>-1$, then the spectral resolution associated with the α-shifted Jacobi equation is given by

$$G(x)=\sum_{m=0}^{\infty}\frac{m!\,\Gamma(1-\alpha+\beta+2m)\,\Gamma(1-\alpha+\beta+m)\{(1-x)^{-\alpha}P_m^{(-\alpha,\beta)}(x)\}}{2^{1-\alpha+\beta}\Gamma(1-\alpha+m)\,\Gamma(1+\beta+m)}$$

$$\times\int_{-1}^{1}\{(1-\eta)^{-\alpha}P_m^{(-\alpha,\beta)}(\eta)\}\,G(\eta)(1-\eta)^{\alpha}(1+\eta)^{\beta}\,d\eta.$$

(2) If $H(x)\in L^2(-1,1;(1-x)^{\alpha}(1+x)^{\beta})$ with $\alpha>-1$, $\beta<1$, then the spectral resolution associated with the β-shifted Jacobi equation is given by

$$H(x)=\sum_{m=0}^{\infty}\frac{m!\,\Gamma(1+\alpha-\beta+2m)\,\Gamma(1+\alpha-\beta+m)\{(1+x)^{-\beta}P_m^{(\alpha,-\beta)}(x)\}}{2^{1+\alpha-\beta}\Gamma(1+\alpha+m)\,\Gamma(1-\beta+m)}$$

$$\times\int_{-1}^{1}\{(1+\eta)^{-\beta}P_m^{(\alpha,-\beta)}(\eta)\}\,H(\eta)(1-\eta)^{\alpha}(1+\eta)^{\beta}\,d\eta.$$

(3) If $I(x) \in L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$ with $\alpha < 1$, $\beta < 1$, then the spectral resolution associated with the $\alpha\beta$-shifted Jacobi equation is given by

$$I(x) = \sum_{m=0}^{\infty} \frac{\begin{pmatrix} m!\, \Gamma(1-\alpha-\beta+2m)\, \Gamma(1-\alpha-\beta+m) \\ \times \{(1-x)^{-\alpha} (1+x)^{-\beta} P_m^{(-\alpha, -\beta)}(x)\} \end{pmatrix}}{2^{1-\alpha-\beta}\Gamma(1-\alpha+m)\, \Gamma(1-\beta+m)}$$
$$\times \int_{-1}^{1} \{(1-\eta)^{-\alpha} (1+\eta)^{-\beta} P_m^{(-\alpha, -\beta)}(\eta)\}\, I(\eta)(1-\eta)^\alpha (1+\eta)^\beta \, d\eta.$$

These expansions, which involve Jacobi polynomials in a new way, are new.

(4) In comparison, if $F(x) \in L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$ with $\alpha > -1$, $\beta > -1$, then the spectral resolution associated with the classical Jacobi equation is given by

$$F(x) = \sum_{n=0}^{\infty} \frac{n!\, \Gamma(1+\alpha+\beta+2n)\, \Gamma(1+\alpha+\beta+n)\, P_n^{(\alpha, \beta)}(x)}{2^{1+\alpha+\beta}\Gamma(1+\alpha+n)\, \Gamma(1+\beta+n)}$$
$$\times \int_{-1}^{1} P_n^{(\alpha, \beta)}(\eta)\, F(\eta)(1-\eta)^\alpha (1+\eta)^\beta \, d\eta.$$

This, of course, is well known.

In the left-definite case, the Jacobi operators when $\alpha < 0$ (not $\alpha < 1$), $\beta > -1$; $\alpha > -1$, $\beta < 0$; $\alpha < 0$, $\beta < 0$; and $\alpha > -1$, $\beta > -1$ are shown to be self-adjoint. It is shown that the spectra for the α-shifted, β-shifted, $\alpha\beta$-shifted, and the classical Jacobi equations are discrete: $\{n(\alpha+\beta+n+1)+1-\alpha\}_{n=0}^{\infty}$, $\{n(\alpha+\beta+n+1)+1-\beta\}_{n=0}^{\infty}$, $\{n(\alpha+\beta+n+1)+1-\alpha-\beta\}_{n=0}^{\infty}$, and $\{n(\alpha+\beta+n+1)+1\}_{n=0}^{\infty}$, with eigenfunctions $\{(1-x)^{-\alpha} P_n^{(-\alpha, \beta)}(x)\}_{n=0}^{\infty}$, $\{(1+x)^{-\beta} P_n^{(-\alpha, \beta)}(x)\}_{n=0}^{\infty}$, $\{(1-x)^{-\alpha} (1+x)^{-\beta} P_n^{(-\alpha, -\beta)}(x)\}_{n=0}^{\infty}$, and $\{P_n^{(\alpha, \beta)}(x)\}_{n=0}^{\infty}$, respectively. Finally we show that the spectral resolutions from $L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$ hold in the new space $H^1(-1, 1; (1-x)^{\alpha+1} (1+x)^{\beta+1}, (1-x)^\alpha (1+x)^\beta)$ as well.

2. The Jacobi Operators under the Right-Definite Norm

The Jacobi differential operator at the point 1 is in the limit circle case when $-1 < \alpha < 1$. It is the limit point when $\alpha \leqslant -1$ and $\alpha \geqslant 1$. At the point -1, the operator is in the limit circle case when $-1 < \beta < 1$. It is limit point when $\beta \leqslant -1$ and $\beta \geqslant 1$. The study of Jacobi's equation in the right-definite case in $L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$ with $\alpha > -1$, $\beta > -1$ can be found in books such as Jackson [5], Szëgo [11], Titchmarsh [12]. The analysis on the Jacobi differential equation in these books fails when $\alpha < -1$, $\beta > -1$; $\alpha > -1$, $\beta < -1$; and $\alpha < -1$; $\beta < -1$ because $P_n^{(\alpha,\beta)}(x)$ is not in $L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$.

A discussion of singular Sturm–Liouville theory may be found in [7, 12].

THEOREM 2.1. *The cases* $\alpha > -1$, $\beta > -1$ *and* $\alpha < 1$, $\beta > -1$ *are equivalent under the transformation*

$$y(x) = (1-x)^{-\alpha} z(x).$$

Proof. Let $y(x) = (1-x)^{-\alpha} z(x)$. Then

$$y'(x) = (1-x)^{-\alpha} z'(x) + \alpha(1-x)^{-\alpha-1} z(x),$$

$$y''(x) = (1-x)^{-\alpha} z''(x) + 2\alpha(1-x)^{-\alpha-1} z'(x) + \alpha(\alpha+1)(1-x)^{-\alpha-2} z(x).$$

Substituting these in the Jacobi differential equation and simplifying, we get

$$\begin{aligned}(1-x^2)\, z'' &+ [\beta + \alpha - (-\alpha + \beta + 2)\, x]\, z' \\ &+ (\alpha + n)(-\alpha + \beta + (\alpha + n) + 1)\, z = 0,\end{aligned}$$

where $\alpha < 1$, $\beta > -1$. Note the change in the eigenvalue.

Let $\alpha = -\hat{\alpha}$ and $\alpha + n = m$. Then we have

$$(1-x^2)\, z'' + [\beta - \hat{\alpha} - (\hat{\alpha} + \beta + 2)\, x]\, z' + m(\hat{\alpha} + \beta + m + 1)\, z = 0,$$

where $\hat{\alpha} > -1$, $\beta > -1$. ∎

THEOREM 2.2. *The cases* $\alpha > -1$, $\beta > -1$ *and* $\alpha > -1$, $\beta < 1$ *are equivalent under the transformation*

$$y(x) = (1+x)^{-\beta} z(x).$$

Proof. Let $y(x) = (1+x)^{-\beta} z(x)$. Then

$$y'(x) = (1+x)^{-\beta} z'(x) - \alpha(1+x)^{-\beta-1} z(x),$$

$$y''(x) = (1+x)^{-\beta} z''(x) - 2\beta(1+x)^{-\beta-1} z'(x) + \beta(\beta+1)(1+x)^{-\beta-2} z(x).$$

Substituting these in the Jacobi differential equation and simplifying, we get

$$\begin{aligned}(1-x^2)\, z'' &+ [-\beta - \alpha - (\alpha - \beta + 2)\, x]\, z' \\ &+ (\beta + n)(\alpha - \beta + (\beta + n) + 1)\, z = 0,\end{aligned}$$

where $\alpha > -1$, $\beta < 1$. Again, note the change in the eigenvalue.

Let $\beta = -\check{\beta}$ and $\beta + n = m$. Then we have

$$(1-x^2)\,z'' + [\check{\beta} - \alpha - (\alpha + \check{\beta} + 2)\,x]\,z' + m(\alpha + \check{\beta} + m + 1)\,z = 0,$$

where $\alpha > -1$, $\check{\beta} > -1$. ∎

THEOREM 2.3. *The cases* $\alpha > -1$, $\beta > -1$ *and* $\alpha < 1$, $\beta < 1$ *are equivalent under the transformation*

$$y(x) = (1-x)^{-\alpha}\,(1+x)^{-\beta}\,z(x).$$

Proof. Let $y(x) = (1-x)^{-\alpha}\,(1+x)^{-\beta}\,z(x)$. Then

$$\begin{aligned} y'(x) &= (1-x)^{-\alpha}\,(1+x)^{-\beta}\,z'(x) - \beta(1+x)^{-\beta-1}\,(1-x)^{-\alpha}\,z(x) \\ &\quad + \alpha(1-x)^{-\alpha-1}\,(1+x)^{-\beta}\,z(x), \\ y''(x) &= (1-x)^{-\alpha}\,(1+x)^{-\beta}\,z''(x) \\ &\quad + [2\alpha(1-x)^{-\alpha-1}\,(1+x)^{-\beta} - 2\beta(1-x)^{-\alpha}\,(1+x)^{-\beta-1}] \\ &\quad + [\alpha(\alpha+1)(1-x)^{-\alpha-2}\,(1+x)^{-\beta} + \beta(\beta+1)(1-x)^{-\alpha}\,(1+x)^{-\beta-2} \\ &\quad - 2\alpha\beta(1-x)^{-\alpha-1}\,(1+x)^{-\beta-1}]\,z(x). \end{aligned}$$

Substituting these in the Jacobi differential equation and simplifying, we get

$$\begin{aligned} &(1-x^2)\,z'' + [-\beta + \alpha - (-\alpha - \beta + 2)\,x]\,z' \\ &\qquad + (\alpha + \beta + n)(-\alpha - \beta + (\alpha + \beta + n) + 1)\,z = 0, \end{aligned}$$

where $\alpha < 1$, $\beta < 1$. Here the eigenvalue has been doubly shifted.

Let $\alpha = -\tilde{\alpha}$, $\beta = -\tilde{\beta}$, and $\alpha + \beta + n = m$. Then we have

$$(1-x^2)\,z'' + [\tilde{\beta} - \tilde{\alpha} - (\tilde{\alpha} + \tilde{\beta} + 2)\,x]\,z' + m(\tilde{\alpha} + \tilde{\beta} + m + 1)\,z = 0,$$

Here $\tilde{\alpha} > -1$, $\tilde{\beta} > -1$. ∎

DEFINITION 2.1. We call the equation

$$(1-x^2)\,y'' + [\beta - \alpha - (\alpha + \beta + 2)\,x]\,y' + n(\alpha + \beta + n + 1)\,y = 0,$$

where $\alpha > -1$, $\beta > -1$ the classical Jacobi equation, the equation

$$\begin{aligned} &(1-x^2)\,z'' + [\beta + \alpha - (-\alpha + \beta + 2)\,x]\,z' \\ &\qquad + (\alpha + n)(-\alpha + \beta + (\alpha + n) + 1)\,z = 0, \end{aligned}$$

where $\alpha < 1$, $\beta > -1$ the α-shifted Jacobi equation, the equation

$$(1-x^2)\,z'' + [-\beta - \alpha - (\alpha - \beta + 2)\,x]\,z' + (\beta + n)(\alpha - \beta + (\beta + n) + 1)\,z = 0,$$

where $\alpha > -1$, $\beta < 1$ the β-shifted Jacobi equation, the equation

$$(1-x^2)\,z'' + [-\beta + \alpha - (-\alpha - \beta + 2)\,x]\,z' + (\alpha + \beta + n)(-\alpha - \beta + (\alpha + \beta + n) + 1)\,z = 0,$$

where $\alpha < 1$, $\beta < 1$ the $\alpha\beta$-shifted Jacobi equation.

First, the classical Jacobi equation is considered. The boundary conditions at 1 and -1, for the classical Jacobi equation can be found using Wronskian limits. (See [7].)

DEFINITION 2.2. Let D_{L_C} denote those functions y with the following properties:

(1) y is in $L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$.

(2) $ly = [-((1-x)^{\alpha+1}(1+x)^{\beta+1}y')' + (1-x)^\alpha (1+x)^\beta y]/(1-x)^\alpha (1+x)^\beta$ exists a.e. and is in $L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$.

(3) $-\lim_{x \to \pm 1}(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x) = 0$, $\alpha > -1$, $\beta > -1$.

We define the operator L_C by setting $L_C y = ly$ for all y in D_{L_C}.

The boundary conditions $\lim_{x \to \pm 1}(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x) = 0$ for the classical Jacobi equation are automatically satisfied if $\alpha \geqslant 1$ and $\beta \geqslant 1$, respectively.

During the past 15 years there has been a substantial increase in the knowledge of singular boundary conditions [7]. Condition (3) in Definition 2.2 is the limit of Wronskians involving y and 1. It is such Wronskians that generate singular boundary conditions. They are generalizations of regular conditions which involve y and y' at end points, and they always exist in the L^2 setting.

Kaper, Kwong, and Zettl [6] has shown that in many instances conditions (3) are equivalent to assuming that y has a finite limit at end points, a condition sometimes used instead. It is sometimes applicable in our situation, sometimes not.

THEOREM 2.4. *L_C is self-adjoint.*

If $F(x) \in L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$ and $y \in D_{L_C}$, then the spectral resolutions of the classical Jacobi equation when $\alpha > -1$, $\beta > -1$ are given by

$$F(x)=\sum_{n=0}^{\infty}\frac{n!\,\Gamma(1+\alpha+\beta+2n)\,\Gamma(1+\alpha+\beta+n)\,P_n^{(\alpha,\beta)}(x)}{2^{1+\alpha+\beta}\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}$$
$$\times\int_{-1}^{1}P_n^{(\alpha,\beta)}(\eta)\,F(\eta)(1-\eta)^{\alpha}\,(1+\eta)^{\beta}\,d\eta,$$

$$L_C y=\sum_{n=0}^{\infty}[n(\alpha+\beta+n+1)+1]$$
$$\times\frac{n!\,\Gamma(1+\alpha+\beta+2n)\,\Gamma(1+\alpha+\beta+n)\,P_n^{(\alpha,\beta)}(x)}{2^{1+\alpha+\beta}\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}$$
$$\times\int_{-1}^{1}P_n^{(\alpha,\beta)}(\eta)\,y(\eta)(1-\eta)^{\alpha}\,(1+\eta)^{\beta}\,d\eta,$$

$$(L_C-\lambda)^{-1}F=\sum_{n=0}^{\infty}\frac{1}{[n(\alpha+\beta+n+1)+1]-\lambda}$$
$$\times\frac{n!\,\Gamma(1+\alpha+\beta+2n)\,\Gamma(1+\alpha+\beta+n)\,P_n^{(\alpha,\beta)}(x)}{2^{1+\alpha+\beta}\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}$$
$$\times\int_{-1}^{1}P_n^{(\alpha,\beta)}(\eta)\,F(\eta)(1-\eta)^{\alpha}\,(1+\eta)^{\beta}\,d\eta.$$

These results are well known.

Second, we consider the α-shifted Jacobi equation. The boundary conditions at -1 and 1, for the α-shifted Jacobi equation can also be found using the Wronskian limits.

DEFINITION 2.3. Let $D_{\mathfrak{L}_S}$ denote those functions z with the following properties:

(1) z is in $L^2(-1,1;(1-x)^{\hat{\alpha}}\,(1+x)^{\beta})$.

(2) $lz=[-((1-x)^{\hat{\alpha}+1}\,(1+x)^{\beta+1}\,z')'+(1-x)^{\hat{\alpha}}\,(1+x)^{\beta}\,z]/(1-x)^{\hat{\alpha}}\,(1+x)^{\beta}$ exists a.e. and is in $L^2(-1,1;(1-x)^{\hat{\alpha}}\,(1+x)^{\beta})$.

(3) $-\lim_{x\to\pm1}(1-x)^{\hat{\alpha}+1}\,(1+x)^{\beta+1}\,z'(x)=0$, $\hat{\alpha}>-1$, $\beta>-1$.

We define the operator $\hat{\mathfrak{L}}_S$ by setting $\hat{\mathfrak{L}}_S z=lz$ for all z in $D_{\mathfrak{L}_S}$.

We recall that $z(x)=(1-x)^{\alpha}\,y(x)$ and $z'(x)=(1-x)^{\alpha}\,y'(x)-\alpha(1-x)^{\alpha-1}\,y(x)$, so the boundary conditions $-\lim_{x\to\pm1}(1-x)^{\hat{\alpha}+1}\,(1+x)^{\beta+1}\,z'(x)=0$ are equivalent to

$$-\lim_{x\to\pm1}(1+x)^{\beta+1}\,[(1-x)\,y'(x)-\alpha y(x)]=0.$$

Therefore we can restate the last definition.

DEFINITION 2.4. Let $D_{\hat{L}_S}$ denote those functions y with the following properties:

(1) y is in $L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$.

(2) $ly = [-((1-x)^{\alpha+1} (1+x)^{\beta+1} y')' + (1-x)^\alpha (1+x)^\beta y]/(1-x)^\alpha (1+x)^\beta$ exists a.e. and is in $L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$.

(3) $-\lim_{x \to \pm 1}(1+x)^{\beta+1} [(1-x) y'(x) - \alpha y(x)] = 0$, $\alpha < 1$, $\beta > -1$.

We define the operator $\hat{L}_S$ by setting $\hat{L}_S y = ly$ for all y in $D_{\hat{L}_S}$.

THEOREM 2.5. *$\hat{L}_S$ is self-adjoint.*

If $G(x) \in L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$ and $y \in D_{\hat{L}_S}$, then the spectral resolutions of the α-shifted Jacobi equation when $\alpha < 1$, $\beta > -1$ are given by

$$G(x) = \sum_{m=0}^{\infty} \frac{\begin{pmatrix} m!\, \Gamma(1-\alpha+\beta+2m)\, \Gamma(1-\alpha+\beta+m) \\ \times \{(1-x)^{-\alpha} P_m^{(-\alpha,\beta)}(x)\} \end{pmatrix}}{2^{1-\alpha+\beta} \Gamma(1-\alpha+m)\, \Gamma(1+\beta+m)} \times \int_{-1}^{1} \{(1-\eta)^{-\alpha} P_m^{(-\alpha,\beta)}(\eta)\}\, G(\eta)(1-\eta)^\alpha (1+\eta)^\beta \, d\eta,$$

$$\hat{L}_S y = \sum_{m=0}^{\infty} [m(\alpha+\beta+m+1) + 1 - \alpha] \times \frac{\begin{pmatrix} m!\, \Gamma(1-\alpha+\beta+2m)\, \Gamma(1-\alpha+\beta+m) \\ \times \{(1-x)^{-\alpha} P_m^{(-\alpha,\beta)}(x)\} \end{pmatrix}}{2^{1-\alpha+\beta} \Gamma(1-\alpha+m)\, \Gamma(1+\beta+m)} \times \int_{-1}^{1} \{(1-\eta)^{-\alpha} P_m^{(-\alpha,\beta)}(\eta)\}\, y(\eta)(1-\eta)^\alpha (1+\eta)^\beta \, d\eta,$$

$$(\hat{L}_S - \lambda)^{-1} G = \sum_{m=0}^{\infty} \frac{1}{[m(\alpha+\beta+m+1) + 1 - \alpha] - \lambda} \times \frac{\begin{pmatrix} m!\, \Gamma(1-\alpha+\beta+2m)\, \Gamma(1-\alpha+\beta+m) \\ \times \{(1-x)^{-\alpha} P_m^{(-\alpha,\beta)}(x)\} \end{pmatrix}}{2^{1-\alpha+\beta} \Gamma(1-\alpha+m)\, \Gamma(1+\beta+m)} \times \int_{-1}^{1} \{(1-\eta)^{-\alpha} P_m^{(-\alpha,\beta)}(\eta)\}\, G(\eta)(1-\eta)^\alpha (1+\eta)^\beta \, d\eta.$$

This is new. Note the shifted eigenvalues in the second and third expansions.

Third, the β-shifted Jacobi equation is considered.

DEFINITION 2.5. Let $D_{\tilde{L}_S}$ denote those functions z with the following properties:

(1) z is in $L^2(-1, 1; (1-x)^\alpha (1+x)^{\breve{\beta}})$.

(2) $lz = [-((1-x)^{\alpha+1} (1+x)^{\breve{\beta}+1} z')' + (1-x)^\alpha (1+x)^{\breve{\beta}} z]/(1-x)^\alpha (1+x)^{\breve{\beta}}$ exists a.e. and is in $L^2(-1, 1; (1-x)^\alpha (1+x)^{\breve{\beta}})$.

(3) $-\lim_{x \to \pm 1} (1-x)^{\alpha+1} (1+x)^{\breve{\beta}+1} z'(x) = 0$, $\alpha > -1$, $\breve{\beta} > -1$.

We define the operator $\breve{\mathbb{L}}_S$ by setting $\breve{\mathbb{L}}_S z = lz$ for all z in $D_{\breve{\mathbb{L}}_S}$.

We recall that $z(x) = (1+x)^\beta y(x)$ and $z'(x) = (1+x)^\beta y'(x) + \beta(1+x)^{\beta-1} y(x)$, so the boundary conditions $-\lim_{x \to \pm 1} (1-x)^{\alpha+1} (1+x)^{\breve{\beta}+1} z'(x) = 0$ are equivalent to

$$-\lim_{x \to \pm 1} (1-x)^{\alpha+1} [(1+x) y'(x) + \beta y(x)] = 0.$$

Therefore we can restate the last definition.

DEFINITION 2.6. Let $D_{\breve{L}_S}$ denote those functions y with the following properties:

(1) y is in $L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$.

(2) $ly = [-((1-x)^{\alpha+1} (1+x)^{\beta+1} y')' + (1-x)^\alpha (1+x)^\beta y]/(1-x)^\alpha (1+x)^\beta$ exists a.e. and is in $L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$.

(3) $-\lim_{x \to \pm 1} (1-x)^{\alpha+1} [(1+x) y'(x) + \beta y(x)] = 0$, $\alpha > -1$, $\beta < 1$.

We define the operator $\breve{L}_S$ by setting $\breve{L}_S y = ly$ for all y in $D_{\breve{L}_S}$.

THEOREM 2.6. *$\breve{L}_S$ is self-adjoint.*

If $H(x) \in L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$ and $y \in D_{\breve{L}_S}$, then the spectral resolutions of the β-shifted Jacobi equation when $\alpha > -1$, $\beta < 1$ are given by

$$H(x) = \sum_{m=0}^{\infty} \frac{\begin{pmatrix} m!\, \Gamma(1+\alpha-\beta+2m)\, \Gamma(1+\alpha-\beta+m) \\ \{(1+x)^{-\beta} P_m^{(\alpha, -\beta)}(x) \end{pmatrix}}{2^{1+\alpha-\beta} \Gamma(1+\alpha+m)\, \Gamma(1-\beta+m)}$$

$$\times \int_{-1}^{1} \{(1+\eta)^{-\beta} P_m^{(\alpha, -\beta)}(\eta)\}\, H(\eta)(1-\eta)^\alpha (1+\eta)^\beta \, d\eta,$$

$$\breve{L}_S y = \sum_{m=0}^{\infty} [m(\alpha+\beta+m+1) + 1 - \beta]$$

$$\times \frac{\begin{pmatrix} m!\, \Gamma(1+\alpha-\beta+2m)\, \Gamma(1+\alpha-\beta+m) \\ \times \{(1+x)^{-\beta} P_m^{(\alpha, -\beta)}(x)\} \end{pmatrix}}{2^{1+\alpha-\beta} \Gamma(1+\alpha+m)\, \Gamma(1-\beta+m)}$$

$$\times \int_{1}^{1} \{(1+\eta)^{-\beta} P_m^{(\alpha, -\beta)}(\eta)\}\, y(\eta)(1-\eta)^\alpha (1+\eta)^\beta \, d\eta,$$

$$(\check{L}_S-\lambda)^{-1} H=\sum_{m=0}^{\infty} \frac{1}{[m(\alpha+\beta+m+1)+1-\beta]-\lambda}$$
$$\times \frac{\begin{pmatrix} m!\,\Gamma(1+\alpha-\beta+2m)\,\Gamma(1+\alpha-\beta+m) \\ \times\{(1+x)^{-\beta} P_m^{(\alpha,-\beta)}(x)\} \end{pmatrix}}{2^{1+\alpha-\beta}\Gamma(1+\alpha+m)\,\Gamma(1-\beta+m)}$$
$$\times \int_{-1}^{1} \{(1+\eta)^{-\beta} P_m^{(\alpha,-\beta)}(\eta)\}\, H(\eta)(1-\eta)^{\alpha}(1+\eta)^{\beta}\, d\eta.$$

Likewise this is new. Again note the shifted eigenvalues in the second and third expansions.

Finally, we consider the $\alpha\beta$-shifted Jacobi differential equation.

DEFINITION 2.7. Let $D_{\tilde{\mathbb{L}}_S}$ denote those functions z with the following properties:

(1) z is in $L^2(-1, 1; (1-x)^{\tilde{\alpha}}(1+x)^{\tilde{\beta}})$.

(2) $lz=[-((1-x)^{\tilde{\alpha}+1}(1+x)^{\tilde{\beta}+1} z')'+(1-x)^{\tilde{\alpha}}(1+x)^{\tilde{\beta}} z]/(1-x)^{\tilde{\alpha}}(1+x)^{\tilde{\beta}}$ exists a.e. and is in $L^2(-1, 1; (1-x)^{\tilde{\alpha}}(1+x)^{\tilde{\beta}})$.

(3) $-\lim_{x\to\pm 1}(1-x)^{\tilde{\alpha}+1}(1+x)^{\tilde{\beta}+1} z'(x)=0$, $\tilde{\alpha}>-1$, $\tilde{\beta}>-1$.

We define the operator $\tilde{\mathbb{L}}_S$ by setting $\tilde{\mathbb{L}}_S z=lz$ for all z in $D_{\tilde{\mathbb{L}}_S}$.

We recall that $z(x)=(1-x)^{\alpha}(1+x)^{\beta} y(x)$ and its derivative

$$z'(x)=(1-x)^{\alpha}(1+x)^{\beta} y'(x)+[\beta(1-x)^{\alpha}(1+x)^{\beta-1} - \alpha(1-x)^{\alpha-1}(1+x)^{\beta}]\, y(x),$$

so the boundary conditions $-\lim_{x\to\pm 1}(1-x)^{\tilde{\alpha}+1}(1+x)^{\tilde{\beta}+1} z'(x)=0$ are equivalent to

$$-\lim_{x\to\pm 1}[(1-x^2)\,y'(x)+(\beta(1-x)-\alpha(1+x))\,y(x)]=0.$$

Therefore we can restate the last definition.

DEFINITION 2.8. Let $D_{\tilde{L}_S}$ denote those functions y with the following properties:

(1) y is in $L^2(-1, 1; (1-x)^{\alpha}(1+x)^{\beta})$.

(2) $ly=[-((1-x)^{\alpha+1}(1+x)^{\beta+1} y')'+(1-x)^{\alpha}(1+x)^{\beta} y]/(1-x)^{\alpha}(1+x)^{\beta}$ exists a.e. and is in $L^2(-1, 1; (1-x)^{\alpha}(1+x)^{\beta})$.

(3) $-\lim_{x\to\pm 1}[(1-x^2)\,y'(x)+(\beta(1-x)-\alpha(1+x))\,y(x)]=0$, $\alpha<1$, $\beta<1$.

We define the operator $\tilde{L}_S$ by setting $\tilde{L}_S y=ly$ for all y in $D_{\tilde{L}_S}$.

THEOREM 2.7. *$\tilde{L}_S$ is self-adjoint.*

If $I(x) \in L^2(-1, 1; (1-x)^\alpha (1+x)^\beta)$ and $y \in D_{\tilde{L}_S}$, then the spectral resolutions of the $\alpha\beta$-shifted Jacobi equation when $\alpha < 1$, $\beta < 1$ are given by

$$
\begin{aligned}
I(x) = \sum_{m=0}^{\infty} & \frac{\begin{pmatrix} m!\, \Gamma(1-\alpha-\beta+2m)\, \Gamma(1-\alpha-\beta+m) \\ \{(1-x)^{-\alpha} (1+x)^{-\beta} P_m^{(-\alpha,-\beta)}(x)\} \end{pmatrix}}{2^{1-\alpha-\beta}\Gamma(1-\alpha+m)\, \Gamma(1-\beta+m)} \\
& \times \int_{-1}^{1} \{(1-\eta)^{-\alpha} (1+\eta)^{-\beta} P_m^{(-\alpha,-\beta)}(\eta)\} \\
& \times I(\eta)(1-\eta)^\alpha (1+\eta)^\beta \, d\eta, \\
\tilde{L}_S y = \sum_{m=0}^{\infty} & [m(\alpha+\beta+m+1)+1-\alpha-\beta] \\
& \times \frac{\begin{pmatrix} m!\, \Gamma(1-\alpha-\beta+2m)\, \Gamma(1-\alpha-\beta+m) \\ \times \{(1-x)^{-\alpha} (1+x)^{-\beta} P_m^{(-\alpha,-\beta)}(x)\} \end{pmatrix}}{2^{1-\alpha-\beta}\Gamma(1-\alpha+m)\, \Gamma(1-\beta+m)} \\
& \times \int_{-1}^{1} \{(1-\eta)^{-\alpha} (1+\eta)^{-\beta} P_m^{(\alpha,-\beta)}(\eta)\} \\
& \times y(\eta)(1-\eta)^\alpha (1+\eta)^\beta \, d\eta, \\
(\tilde{L}_S - \lambda)^{-1} I = \sum_{m=0}^{\infty} & \frac{1}{[m(\alpha+\beta+m+1)+1-\alpha-\beta]-\lambda} \\
& \times \frac{\begin{pmatrix} m!\, \Gamma(1-\alpha-\beta+2m)\, \Gamma(1-\alpha-\beta+m) \\ \times \{(1-x)^{-\alpha} (1+x)^{-\beta} P_m^{(-\alpha,-\beta)}(x)\} \end{pmatrix}}{2^{1-\alpha-\beta}\Gamma(1-\alpha+m)\, \Gamma(1-\beta+m)} \\
& \times \int_{-1}^{1} \{(1-\eta)^{-\alpha} (1+\eta)^{-\beta} P_m^{(-\alpha,-\beta)}(\eta)\} \\
& \times I(\eta)(1-\eta)^\alpha (1+\eta)^\beta \, d\eta.
\end{aligned}
$$

This doubly shifted expansion is perhaps the most interesting. The eigenvalues are doubly shifted.

3. THE JACOBI OPERATORS UNDER THE LEFT-DEFINITE ENERGY NORM

Left-definite settings for ordinary boundary value problems appeared first in an article [13] by H. Weyl in 1910. His work was followed by A. Pleijel [9, 10], Everitt and Wray [4], Everitt [1], Everitt and Littlejohn [2], and Everitt, Littlejohn, and Krall [3].

They are useful in deriving information concerning convergence of eigenfunction expansions not only for arbitrary elements in $L^2(a, b; w)$, the traditional setting, but also for their derivatives in natural $H^1(a, b; p, q)$ setting.

There has always been difficulty with showing that in the left-definite setting the differential operator remains self-adjoint, and it was not until Krall [8], that the necessary extended Green's formulas were found which allowed self-adjointness to be proved in the regular case.

We adapt the theory of the regular left-definite boundary value problems [8] to the Jacobi equations. One immediate difficulty encountered is that not all elements are in $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$. Therefore we examine asymptotic forms of solutions to determine which are. By the Frobenius method, we find that

$$y_1(x) \approx 1, \qquad y_2(x) \approx (x-1)^{-\alpha},$$

$$y_1'(x) \approx \frac{1}{\alpha+1}, \qquad y_2'(x) \approx -\alpha(x-1)^{-\alpha-1},$$

as $x \to 1$, and

$$y_1(x) \approx 1, \qquad y_2(x) \approx (x+1)^{-\beta},$$

$$y_1'(x) \approx \frac{1}{\beta+1}, \qquad y_2'(x) \approx -\beta(x+1)^{-\beta-1},$$

as $x \to -1$.

It is easy to show that near 1 when $\alpha > 1$ only $y_1(x)$ is in H^1. When $0 < \alpha < 1$, only $y_1(x)$ is in H^1. When $-1 < \alpha < 0$ both $y_1(x)$ and $y_2(x)$ are in H^1. When $\alpha < -1$ only $y_2(x)$ is in H^1.

Similarly, near -1 when $\beta > 1$ only $y_1(x)$ is in H^1. When $0 < \beta < 1$, only $y_1(x)$ is in H^1. When $-1 < \beta < 0$ both $y_1(x)$ and $y_2(x)$ are in H^1. When $\beta < -1$ only $y_2(x)$ is in H^1. Therefore we make the following definitions for the classical and the shifted Jacobi equations in $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$ which are similar to those in $L^2(-1, 1; (1-x)^{\alpha}(1+x)^{\beta})$.

Definition 3.1. We call the equation

$$(1-x^2)\, y''[\beta - \alpha - (\alpha+\beta+2)\, x]\, y' + n[\alpha+\beta+n+1)\, y = 0,$$

where $\alpha > -1$, $\beta > -1$ the classical Jacobi equation in H^1, the equation

$$\begin{aligned}(1-x^2)\, z'' + [\beta + \alpha - (-\alpha+\beta+2)\, x]\, z' \\ + (\alpha+n)(-\alpha+\beta+(\alpha+n)+1)\, z = 0,\end{aligned}$$

where $\alpha<0$, $\beta>-1$ the α-shifted Jacobi equation in H^1, the equation

$$\begin{aligned}(1-x^2)\,z''+[-\beta-\alpha-(\alpha-\beta+2)\,x]\,z'\\ +(\beta+n)(\alpha-\beta+(\beta+n)+1)\,z=0,\end{aligned}$$

where $\alpha>-1$, $\beta<0$ the β-shifted Jacobi equation in H^1, the equation

$$\begin{aligned}(1-x^2)\,z''+[-\beta+\alpha-(-\alpha-\beta+2)\,x]\,z'\\ +(\alpha+\beta+n)(-\alpha-\beta+(\alpha+\beta+n)+1)\,z=0,\end{aligned}$$

where $\alpha<0$, $\beta<0$ the $\alpha\beta$-shifted Jacobi equation in H^1.

The key to extending the $L^2(-1,1;(1-x)^\alpha(1+x)^\beta)$ theory of the Jacobi operator into $H^1(-1,1;(1-x)^{\alpha+1}(1+x)^{\beta+1},(1-x)^\alpha(1+x)^\beta)$ is an extension of the traditional Green's formulas. In order to consider these extensions, we have to show that in the Green's formulas the boundary terms vanish in the classical, α-shifted, β-shifted, and $\alpha\beta$-shifted Jacobi equations.

First, we consider the classical Jacobi equation.

THEOREM 3.1. *If y satisfies*

$$\begin{aligned}&-[(1-x)^{\alpha+1}(1+x)^{\beta+1}y']'+(1-x)^\alpha(1+x)^\beta y\\ &\quad=(1-x)^\alpha(1+x)^\beta f,\qquad \alpha>-1,\beta>-1,\\ &\quad\lim_{x\to\pm1}(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)=0,\end{aligned}$$

and y and f are in $H^1(-1,1;(1-x)^{\alpha+1}(1+x)^{\beta+1},(1-x)^\alpha(1+x)^\beta)$, then

$$\lim_{x\to\pm1}(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)\,\bar{f}(x)=0.$$

Proof. If we solve the nonhomogeneous equation

$$\begin{aligned}&-[(1-x)^{\alpha+1}(1+x)^{\beta+1}y']'+(1-x)^\alpha(1+x)^\beta y\\ &\quad=(1-x)^\alpha(1+x)^\beta F,\qquad \alpha>-1,\beta>-1,\end{aligned}$$

subject to the constraints $\lim_{x\to\pm1}(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)=0$, variation of parameters yield

$$y(x)=-\psi_1(x)\int_{-1}^{x}\frac{\psi_{-1}(\zeta)\,F(\zeta)\,w(\zeta)}{\mathscr{W}}\,d\zeta-\psi_{-1}(x)\int_{x}^{1}\frac{\psi_1(\zeta)\,F(\zeta)\,w(\zeta)}{\mathscr{W}}\,d\zeta,$$

where $\psi_{-1}(x) \approx y_1(x)$ is in $L^2(-1, 0; (1-x)^\alpha (1+x)^\beta)$, $\psi_1(x) \approx y_2(x)$ is in the space $L^2(0, 1; (1-x)^\alpha (1+x)^\beta)$, and $\mathscr{W} = (1-x)^{\alpha+1} (1+x)^{\beta+1} [\psi_{-1}\psi_1' - \psi_{-1}'\psi_1]$ is constant. We also have that

$$y'(x) = -\psi_1'(x) \int_{-1}^{x} \frac{\psi_{-1}(\zeta)\, F(\zeta)\, w(\zeta)}{\mathscr{W}}\, d\zeta - \psi_{-1}'(x) \int_{x}^{1} \frac{\psi_1(\zeta)\, F(\zeta)\, w(\zeta)}{\mathscr{W}}\, d\zeta$$

satisfies $\lim_{x \to \pm 1}(1-x)^{\alpha+1} (1+x)^{\beta+1} y'(x) = 0$.

Near $x = 1$, the derivative becomes

$$y'(x) \approx \alpha(1-x)^{-\alpha-1} \int_{-1}^{x} \frac{\psi_{-1}(\zeta)\, F(\zeta)\, w(\zeta)}{\mathscr{W}}\, d\zeta - \frac{1}{\alpha} \int_{x}^{1} \frac{\psi_1(\zeta)\, F(\zeta)\, w(\zeta)}{\mathscr{W}}\, d\zeta.$$

Since near $x = 1$, the first term of the above approximation is

$$\begin{aligned}
&\left| \alpha(1-x)^{-\alpha-1} \int_{-1}^{x} \frac{\psi_{-1}(\zeta)\, F(\zeta)(1-\zeta)^\alpha (1+\zeta)^\beta}{\mathscr{W}}\, d\zeta \right| \\
&\quad \leqslant (\alpha(x-1)^{-\alpha-1}) \left(\frac{1}{\mathscr{W}}\right) \left(\int_{-1}^{x} |\psi_{-1}|\, (1-\zeta)^\alpha (1+\zeta)^\beta\, d\zeta \right)^{1/2} \\
&\quad \times \left(\int_{-1}^{x} |F|\, (1-\zeta)^\alpha (1+\zeta)^\beta\, d\zeta \right)^{1/2}, \\
&\quad \approx k(1-x)^{-\alpha-1} \left(\int_{-1}^{x} 1(1-\zeta)^\alpha\, d\zeta \right)^{1/2}, \\
&\quad \approx k(1-x)^{-\alpha-1} \left(\frac{(1-x)^{\alpha+1}}{\alpha+1} \right)^{1/2}, \\
&\quad \approx k(1-x)^{-\alpha-1} (1-x)^{(\alpha+1)/2} \approx k(1-x)^{(-\alpha-1)/2},
\end{aligned}$$

and the second term becomes

$$\begin{aligned}
&\left| -\frac{1}{\alpha+1} \int_{x}^{1} \frac{\psi_1(\zeta)\, F(\zeta)(1-\zeta)^\alpha (1+\zeta)^\beta}{\mathscr{W}}\, d\zeta \right| \\
&\quad \leqslant \left(-\frac{1}{\alpha+1} \right) \left(\frac{1}{\mathscr{W}}\right) \left(\int_{x}^{1} |\psi_1|\, (1-\zeta)^\alpha (1+\zeta)^\beta\, d\zeta \right)^{1/2} \\
&\quad \times \left(\int_{x}^{1} |F|\, (1-\zeta)^\alpha (1+\zeta)^\beta\, d\zeta \right)^{1/2}, \\
&\quad \approx \tilde{k} \left(\int_{x}^{1} (\zeta-1)^{-2\alpha} (1-\zeta)^\alpha\, d\zeta \right)^{1/2}, \\
&\quad \approx \tilde{k} \left(\int_{x}^{1} (1-\zeta)^{-\alpha}\, d\zeta \right)^{1/2} \approx \tilde{k}(1-x)(-\alpha+1)/2,
\end{aligned}$$

then we find that $y'(x) \approx k(1-x)^{(-\alpha-1)/2}$.

Now let f be in $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$. Then $(1-x)^{\alpha+1}|f'|^2$ is integrable. This implies that

$$(1-x)^{\alpha+1}|f'|^2 < \frac{k}{(1-x)},$$
$$(1-x)^{(\alpha+1)/2}|f'| < k(1-x)^{-1/2},$$
$$|f'| < k(1-x)^{-\alpha/2-1},$$

and

$$|f| < k(1-x)^{-\alpha/2}.$$

Putting the two estimates together, we have

$$|(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)\bar{f}(x)| < k(1-x)^{\alpha+1}(1+x)^{(-\alpha-1)/2}(1-x)^{-\alpha/2},$$
$$\approx k(1-x)^{1/2},$$

which has limit 0 as $x \to 1$.

A similar discussion holds near $x = -1$. ∎

In order to proceed further, we make the following definition.

Definition 3.2. We denote by $\mathscr{D}_{\mathscr{L}_C}$ those functions y in H^1 satisfying:

(1) y is absolutely continuous on every closed subinterval of $[-1, 1]$.

(2) $(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)$ is absolutely continuous on every closed subinterval of $[-1, 1]$.

(3) $ly = [-((1-x)^{\alpha+1}(1+x)^{\beta+1}y')' + (1-x)^{\alpha}(1+x)^{\beta}y]/(1-x)^{\alpha}(1+x)^{\beta}$ is in H^1.

(4) $\lim_{x\to\pm1}(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x) = 0$, $\alpha > -1$, $\beta > -1$.

We denote by $\mathscr{L}_C$ the differential operator defined by setting $\mathscr{L}_C y = ly$ for all y in $\mathscr{D}_{\mathscr{L}_C}$.

Theorem 3.2. *Let y, and if necessary z, be in $\mathscr{D}_{\mathscr{L}_C}$, then the extended Green's first formula for $\alpha > -1$, $\beta > -1$, is given by*

$$\int_{-1}^{1}(\mathscr{L}_C y)\,\bar{z}(1-x)^{\alpha}(1+x)^{\beta}\,dx$$
$$= \int_{-1}^{1}[(1-x)^{\alpha+1}(1+x)^{\beta+1}y'\bar{z}' + (1-x)^{\alpha}(1+x)^{\beta}y\bar{z}]\,dx,$$

or, in the notation of inner products, $(\mathscr{L}_C y, z)_{L^2} = \langle y, z\rangle_{H^1}$.

Let y and z be in n $\mathscr{D}_{\mathscr{L}_C}$, *then the extended Green's second formula for* $\alpha > -1$, $\beta > -1$ *is given by*

$$\int_{-1}^{1} [(\mathscr{L}_C y)\bar{z} - y(\mathscr{L}_C \bar{z})](1-x)^{\alpha}(1+x)^{\beta}\,dx = 0,$$

or, $(\mathscr{L}_C y, z)_{L^2} - (y, \mathscr{L}_C z)_{L^2} = 0$.

Let y and z be in $\mathscr{D}_{\mathscr{L}_C}$, *then the extended Green's third formula for the parameters* $\alpha > -1$, $\beta > -1$ *is given by*

$$\int_{-1}^{1} (\mathscr{L}_C y)(\mathscr{L}_C \bar{z})(1-x)^{\alpha}(1+x)^{\beta}\,dx$$

$$= \int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1} y'(\mathscr{L}_C \bar{z})' + (1-x)^{\alpha}(1+x)^{\beta} y(\mathscr{L}_C \bar{z})]\,dx,$$

or, $(\mathscr{L}_C y, \mathscr{L}_C z)_{L^2} = \langle y, \mathscr{L}_C z\rangle_{H^1}$.

Let y and z by in $\mathscr{D}_{\mathscr{L}_C}$, *then the extended Green's fourth formula for* $\alpha > -1$, $\beta > -1$ *is given by*

$$\int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1}(\mathscr{L}_C y)'\bar{z}' + (1-x)^{\alpha}(1+x)^{\beta}(\mathscr{L}_C y)\bar{z}]\,dx$$

$$= \int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1} y'(\mathscr{L}_C \bar{z})' + (1-x)^{\alpha}(1+x)^{\beta} y(\mathscr{L}_C \bar{z})]\,dx,$$

or, $\langle \mathscr{L}_C y, z\rangle_{H^1} = \langle y, \mathscr{L}_C z\rangle_{H^1}$.

Proof. To prove the first, insert the expression which is $\mathscr{L}_C y$ and integrate by parts. The limit at -1 and 1 vanishes by Theorem 3.1.

The second follows from the first formula by interchanging y and z and subtracting. We already know this since $H^1 \subset L^2$ and in some sense $\mathscr{L}_C \subset L_C$.

To prove the third, in the first formula replace z by $\mathscr{L}_C z$.

To show the fourth again interchange y and z, this time in the third formula and subtract. ▮

THEOREM 3.3 *$\mathscr{L}_C$ is densely defined in* $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$ *and is symmetric.*

Proof. C_{00}^{∞} functions over $[-1, 1]$ are in $\mathscr{D}_{\mathscr{L}_C}$. These are clearly dense. The extended Green's fourth formula shows $\mathscr{L}_C$ is symmetric. ▮

THEOREM 3.4. *The spectrum of $\mathscr{L}_C$ is discrete, consisting of* $\{n(\alpha+\beta+n+1)+1\}_{n=0}^{\infty}$. *The associated eigenfunctions are* $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, *the classical Jacobi polynomials.*

Proof. Green's first formula shows that

$$\|y_n\|^2_{H^1} = \langle y_n, y_n \rangle = \lambda_n(y_n, y_n) = \{n(\alpha+\beta+n+1)+1\} \|y_n\|^2_{L^2}$$
$$= n(\alpha+\beta+n+1)+1.$$

Hence the normalized eigenfunctions in $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$, when $\alpha > -1$, $\beta > -1$, are $\{P_n^{(\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1}\}_{n=0}^{\infty}$. ∎

THEOREM 3.5. *The resolvent operator* $(\mathscr{L}_C - \lambda)^{-1}$ *exists if* $\lambda \neq n(\alpha+\beta+n+1)+1$, *where* $n = 0, 1, 2, \dots$ *and is a bounded operator. If* λ *is real, then* $(\mathscr{L}_C - \lambda)^{-1}$ *is a bounded, self-adjoint operator. In particular, if* $\lambda = 0$, $\mathscr{L}_C^{-1}$ *exists as a bounded, self-adjoint operator in* $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$.

Proof. Just is in $L^2(-1, 1; (1-x)^{\alpha}(1+x)^{\beta})$, if $(\mathscr{L}_C - \lambda)\,y = F$ *and* F *is arbitrary in the space* $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$, then

$$y = \int_{-1}^{1} G(\lambda, x, \zeta)\, F(\zeta)\, w(\zeta)\, d\zeta = \mathscr{G}_\lambda F,$$

where $G(\lambda, x, \zeta)$ is the Green's function and $\|\mathscr{G}_0\|_{L^2} \leqslant 1$.

Let $\lambda = 0$. Then Green's first formula shows that $\|y_n\|^2_{H^1} = (\mathscr{L}_C\, y, y)$. Substituting $\mathscr{L}_C y = F$, we find

$$\|\mathscr{G}_0 F\|^2_{H^1} \leqslant \|F\|_{L^2} \|\mathscr{G}_0 F\|_{L^2},$$

and

$$\|\mathscr{G}_0 F\|^2_{H^1} \leqslant \|F\|_{L^2} \|\mathscr{G}_0\|_{L^2} \|F\|_{L^2} \leqslant \|\mathscr{G}_0\|_{L^2} \|F\|^2_{H^1}.$$

Hence

$$\frac{\|\mathscr{G}_0 F\|^2_{H^1}}{\|F\|^2_{H^1}} \leqslant \|\mathscr{G}_0\|_{L^2},$$

and

$$\|\mathscr{G}_0\|_{H^1} \leqslant \|\mathscr{G}_0\|^{1/2}_{L^2} \leqslant 1.$$

So $\mathscr{G}_0$ is bounded on $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$. We delay temporarily the proof of the statements concerning $(\mathscr{L}_C - \lambda)^{-1}$. ∎

THEOREM 3.6. $\mathscr{L}_C$ *is a positive, self-adjoint operator on* $\mathscr{D}_{\mathscr{L}_C} \subset H^1$, *with inverse* $\mathscr{G}_0$.

Proof. From the fourth Green's formula we see that $\mathscr{L}_C$ is symmetric. It has equal deficiency indices. Since its range is all of H^1, it is maximally extended. Therefore it is self-adjoint. ∎

The following spectral resolutions hold:

THEOREM 3.7. *If* $F \in H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$, *and* $y \in \mathscr{D}_{\mathscr{L}_C}$, *then*

$$F = \sum_{n=0}^{\infty} \langle F, P_n^{(\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1}\rangle \times (P_n^{(\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1}),$$

$$\mathscr{L}_C y = \sum_{n=0}^{\infty} \{n(\alpha+\beta+n+1)+1\} \times \langle y, P_n^{(\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1}\rangle \times (P_n^{(\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1}),$$

$$(\mathscr{L}_C - \lambda)^{-1} F = \sum_{n=0}^{\infty} \frac{1}{\{n(\alpha+\beta+n+1)+1\} - \lambda} \times \langle F, P_n^{(\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1}\rangle \times (P_n^{(\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1}),$$

where $\alpha > -1, \beta > -1$.

The statements concerning $(\mathscr{L}_C - \lambda)^{-1}$ follow from its spectral resolution. Although these formulas are well known, the H^1 setting is new.

Second, the α-shifted Jacobi equation with $\alpha < 0$, $\beta > -1$, is considered. We recall that if y_1 and y_2 are in $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$, then

$$\langle y_1, y_2\rangle_{H^1} = \int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1} y_1' \bar{y}_2' + (1-x)^{\alpha}(1+x)^{\beta} y_1 \bar{y}_2]\, dx.$$

Integration by parts and transformation $y(x) = (1-x)^{-\alpha} z(x)$ gives

$$\int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1} y_1' \bar{y}_2' + (1-x)^{\alpha}(1+x)^{\beta} y_1 \bar{y}_2]\, dx$$
$$= \int_{-1}^{1} [(1-x)^{-\alpha+1}(1+x)^{\beta+1} z_1' \bar{z}_2' + \{1 - \alpha(\beta+1)\}(1-x)^{-\alpha} \times (1+x)^{\beta} z_1 \bar{z}_2]\, dx + \alpha(1-x)^{-\alpha}(1+x)^{\beta+1} z_1 \bar{z}_2 \big|_{-1}^{1},$$

where $\alpha < 0$, $\beta > -1$.

It is clear that $\lim_{x\to -1}(1-x)^{-\alpha}(1+x)^{\beta+1} z_1(x)\bar{z}_2(x)=0$.
Now we have to show that $\lim_{x\to 1}(1-x)^{-\alpha}(1+x)^{\beta+1} z_1(x)\bar{z}_2(x)=0$.

THEOREM 3.8. *Let* $\lim_{x\to 1}(1-x)^{-\alpha+1}(1+x)^{\beta+1} z_1'(x)=0$ *and* $\lim_{x\to 1}(1-x)^{-\alpha}(1+x)^{\beta+1} z_1(x)z_2(x)=k$, $\alpha<0$, $\beta>-1$. *Then* $k=0$.

Proof. From the previous calculations we know that

$$\lim_{x\to 1}(1-x)^{-\alpha}(1+x)^{\beta+1} z_1^2(x)=k_1,$$

and

$$\lim_{x\to 1}(1-x)^{-\alpha}(1+x)^{\beta+1} z_1^2(x)=k_2.$$

So

$$\lim_{x\to 1}(1-x)^{-\alpha}(1+x)^{\beta+1} z_1(x)z_2(x)=\pm(k_1k_2)^{1/2}.$$

If either k_1 or k_2 is zero, we are done. Assume neither is zero. Then

$$\lim_{x\to 1}(1-x)^{-\alpha}(1+x)^{\beta+1} z_1^2(x)=k_1\neq 0,$$

and

$$\lim_{x\to 1}(1-x)^{-\alpha+1} z_1'(x)=0.$$

Let $z_1=y$.

The last two limits imply that for any $\varepsilon>0$, if x is sufficiently near 0, then

$$-\varepsilon/(1-x)<y'/y^2<\varepsilon/(1-x).$$

Integrate throughout from x to c, both near 0, to find

$$[1/y(c)-\varepsilon ln(1-c)+\varepsilon ln(1-x)]$$
$$<1/y(x)<[1/y(c)+\varepsilon ln(1-c)-ln(1-x)].$$

Taking reciprocals, this implies

$$[1/y(c)-\varepsilon ln(1-c)+\varepsilon ln(1-x)]^{-1}$$
$$<y(x)<[1/y(c)+\varepsilon ln(1-c)-\varepsilon ln(1-x)]^{-1}.$$

As x approaches 1, $ln(1-x)$ approaches $-\infty$, so the left lower bound approaches 0 *through positive values*, while the right upper bound approaches 0 *through negative values*. This cannot happen at the same time, since $y(x)$ is caught in the middle. ∎

We are now ready to exhibit the extended Green's formulas for the α-shifted Jacobi operator.

DEFINITION 3.3. We denote by $\mathscr{D}_{\hat{\mathscr{L}}_S}$ those functions y in H^1 satisfying:

(1) y is absolutely continuous on every closed subinterval of $[-1, 1]$.

(2) $(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(x)$ is absolutely continuous on every closed subinterval of $[-1, 1]$.

(3) $ly = [-((1-x)^{\alpha+1}(1+x)^{\beta+1}y')' + (1-x)^{\alpha}(1+x)^{\beta}y]/(1-x)^{\alpha}(1+x)^{\beta}$ is in H^1.

(4) $\lim_{x\to\pm 1}(1+x)^{\beta+1}[(1-x)y'(x) - \alpha y(x)] = 0$, $\alpha < 0$, $\beta > -1$.

We denote by $\hat{\mathscr{L}}_S$ the differential operator defined by setting $\hat{\mathscr{L}}_S y = ly$ for all y in $\mathscr{D}_{\hat{\mathscr{L}}_S}$.

THEOREM 3.9. *Let y, and if necessary z, be in $\mathscr{D}_{\hat{\mathscr{L}}_S}$, then the extended Green's first formula for $\alpha < 0$, $\beta > -1$, is given by*

$$\int_{-1}^{1} (\hat{\mathscr{L}}_S y)\,\bar{z}(1-x)^{\alpha}(1+x)^{\beta}\,dx$$
$$= \int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1}y'\bar{z}' + (1-x)^{\alpha}(1+x)^{\beta}y\bar{z}]\,dx,$$

or, in the notation of inner products, $(\hat{\mathscr{L}}_S y, z)_{L^2} = \langle y, z\rangle_{H^1}$. Let y and z be in $\mathscr{D}_{\hat{\mathscr{L}}_S}$, then the extended Green's second formula $\alpha < 0$, $\beta > -1$ is given by

$$\int_{-1}^{1} [(\hat{\mathscr{L}}_S y)\,\bar{z} - y(\hat{\mathscr{L}}_S \bar{z})](1-x)^{\alpha}(1+x)^{\beta}\,dx = 0,$$

or, $(\hat{\mathscr{L}}_S y, z)_{L^2} - (y, \hat{\mathscr{L}}_S z)_{L^2} = 0$. Let y and z be in $\mathscr{D}_{\hat{\mathscr{L}}_S}$, then the extended Green's third formula for the parameters $\alpha < 0$, $\beta > -1$ is given by

$$\int_{-1}^{1} (\hat{\mathscr{L}}_S y)(\hat{\mathscr{L}}_S \bar{z})(1-x)^{\alpha}(1+x)^{\beta}\,dx$$
$$= \int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1}y'(\hat{\mathscr{L}}_S \bar{z})' + (1-x)^{\alpha}(1+x)^{\beta}y(\hat{\mathscr{L}}_S \bar{z})]\,dx,$$

or, $(\hat{\mathscr{L}}_S y, \hat{\mathscr{L}}_S z)_{L^2} = \langle y, \hat{\mathscr{L}}_S z\rangle_{H^1}$. Let y and z be in $\mathscr{D}_{\hat{\mathscr{L}}_S}$, then the extended Green's fourth formula for $\alpha < 0$, $\beta > -1$ is given by

$$\int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1}(\hat{\mathscr{L}}_S y)'\bar{z}' + (1-x)^{\alpha}(1+x)^{\beta}(\hat{\mathscr{L}}_S y)\bar{z}]\,dx$$
$$= \int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1} y'(\hat{\mathscr{L}}_S \bar{z})' + (1-x)^{\alpha}(1+x)^{\beta} y(\hat{\mathscr{L}}_S \bar{z})]\,dx,$$

or, $\langle \hat{\mathscr{L}}_S y, z\rangle_{H^1} = \langle y, \hat{\mathscr{L}}_S z\rangle_{H^1}$.

THEOREM 3.10. *$\hat{\mathscr{L}}_S$ is densely defined in $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$ and is symmetric.*

The spectrum of $\hat{\mathscr{L}}_S$ is discrete, consisting of $\{n(\alpha+\beta+n+1)+1-\alpha\}_{n=0}^{\infty}$. The associated eigenfunctions are $\{(1-x)^{-\alpha} P_n^{(-\alpha,\beta)}(x)\}_{n=0}^{\infty}$, the α-shifted Jacobi polynomials.

Hence the normalized eigenfunctions in $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$, when $\alpha < 0$, $\beta > -1$, are $\{(1-x)^{-\alpha} P_n^{(-\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha}\}_{n=0}^{\infty}$.

THEOREM 3.11. *The resolvent operator $(\hat{\mathscr{L}}_S-\lambda)^{-1}$ exists if $\lambda \neq n(\alpha+\beta+n+1)+1-\alpha$, where $n=0, 1, 2, \ldots$ and is a bounded operator. If λ is real, then $(\hat{\mathscr{L}}_S-\lambda)^{-1}$ is a bounded, self-adjoint operator. In particular, if $\lambda=0$, $\hat{\mathscr{L}}_S^{-1}$ exists as a bounded, self-adjoint operator in $H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$.*

THEOREM 3.12. *$\hat{\mathscr{L}}_S$ is a positive, self-adjoint operator on $\mathscr{D}_{\hat{\mathscr{L}}_S} \subset H^1$.*

The following spectral resolutions hold:

THEOREM 3.13. *If $G \in H^1(-1, 1; (1-x)^{\alpha+1}(1+x)^{\beta+1}, (1-x)^{\alpha}(1+x)^{\beta})$, and $y \in \mathscr{D}_{\hat{\mathscr{L}}_S}$, then*

$$G = \sum_{n=0}^{\infty} \langle G, (1-x)^{-\alpha} P_n^{(-\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha}\rangle$$
$$\times ((1-x)^{-\alpha} P_n^{(-\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha}),$$

$$\hat{\mathscr{L}}_S y = \sum_{n=0}^{\infty} \{n(\alpha+\beta+n+1)+1-\alpha\}$$
$$\times \langle y, (1-x)^{-\alpha} P_n^{(-\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha}\rangle$$
$$\times ((1-x)^{-\alpha} P_n^{(-\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha}),$$

$$(\hat{\mathscr{L}}_S-\lambda)^{-1} G = \sum_{n=0}^{\infty} \frac{1}{\{n(\alpha+\beta+n+1)+1-\alpha\}-\lambda}$$
$$\times \langle G, (1-x)^{-\alpha} P_n^{(-\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha}\rangle$$
$$\times ((1-x)^{-\alpha} P_n^{(-\alpha,\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha}),$$

where $\alpha<0$, $\beta>-1$.

The discussion for the β-shifted Jacobi equation is identical to the α-shifted equation except α is substituted for β, and x for $-x$.

Finally, we consider the $\alpha\beta$-shifted Jacobi equation. Again, we recall that

$$\langle y_1, y_2 \rangle_{H^1} = \int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1} y_1' \bar{y}_2' + (1-x)^{\alpha}(1+x)^{\beta} y_1 \bar{y}_2]\, dx.$$

Integration by parts and transformation $y(x) = (1-x)^{-\alpha}(1+x)^{-\beta} z(x)$, gives

$$\begin{aligned}
&\int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1} y_1' \bar{y}_2' + (1-x)^{\alpha}(1+x)^{\beta} y_1 \bar{y}_2]\, dx \\
&\quad = \int_{-1}^{1} [(1-x)^{-\alpha+1}(1+x)^{-\beta+1} z_1' \bar{z}_2' + \{1-(\alpha+\beta)\} \\
&\qquad \times (1-x)^{-\alpha}(1+x)^{-\beta} z_1 \bar{z}_2]\, dx \\
&\qquad + \{\alpha(1-x)^{-\alpha}(1+x)^{-\beta+1} z_1 \bar{z}_2 - \beta(1-x)^{-\alpha+1}(1+x)^{-\beta} z_1 \bar{z}_2\}\big|_{-1}^{1},
\end{aligned}$$

where $\alpha < 0$, $\beta < 0$.

It is clear that

$$\lim_{x \to -1} (1-x)^{-\alpha}(1+x)^{-\beta+1} z_1(x)\, \bar{z}_2(x) = 0,$$

and

$$\lim_{x \to 1} (1-x)^{-\alpha+1}(1+x)^{-\beta} z_1(x)\, \bar{z}_2(x) = 0.$$

Now we have to show that

$$\lim_{x \to 1} (1-x)^{-\alpha}(1+x)^{-\beta+1} z_1(x)\, \bar{z}_2(x) = 0,$$

and

$$\lim_{x \to -1} (1-x)^{-\alpha+1}(1+x)^{-\beta} z_1(x)\, \bar{z}_2(x) = 0.$$

THEOREM 3.14. *Let*

$$\lim_{x \to -1} (1-x)^{-\alpha+1}(1+x)^{-\beta+1} z_1'(x) = 0, \qquad \alpha < 0,\ \beta < 0.$$

Let

$$\lim_{x \to -1} (1-x)^{-\alpha+1}(1+x)^{-\beta} z_1(x)\, z_2(x) = k.$$

Then $k = 0$.

Proof. See Theorem 3.8.

THEOREM 3.15. *Let*

$$\lim_{x \to -1} (1-x)^{-\alpha+1}(1+x)^{-\beta+1} z_1'(x) = 0, \qquad \alpha < 0,\ \beta < 0.$$

Let

$$\lim_{x \to -1} (1-x)^{-\alpha+1}(1+x)^{-\beta} z_1(x)\, z_2(x) = k.$$

Then $k = 0$.

Proof. See Theorem 3.8.

We are now ready to exhibit the extended Green's formulas for the $\alpha\beta$-shifted Jacobi operator.

DEFINITION 3.4. We denote by $\mathscr{D}_{\mathscr{P}_S}$ those functions y in H^1 satisfying:

(1) y is absolutely continuous on every closed subinterval of $[-1, 1]$.

(2) $(1-x)^{\alpha+1}(1+x)^{\beta+1} y'(x)$ is absolutely continuous on every closed subinterval of $[-1, 1]$.

(3) $ly = [-((1-x)^{\alpha+1}(1+x)^{\beta+1} y')' + (1-x)^{\alpha}(1+x)^{\beta} y]/(1-x)^{\alpha}(1+x)^{\beta}$ is in H^1.

(4) $\lim_{x \to \pm 1}[(1-x^2)\, y'(x) + (\beta(1-x) - \alpha(1+x))\, y(x)] = 0, \quad \alpha < 0, \beta < 0.$

We denote by $\tilde{\mathscr{L}}_S$ the differential operator defined by setting $\tilde{\mathscr{L}}_S y = ly$ for all y in $\mathscr{D}_{\mathscr{P}_S}$.

THEOREM 3.16. *Let y, and if necessary z, be in $\mathscr{D}_{\mathscr{P}_S}$, then the extended Green's first formula for $\alpha < 0$, $\beta < 0$, is given by*

$$\int_{-1}^{1} (\tilde{\mathscr{L}}_S y)\, \bar{z}(1-x)^{\alpha}(1+x)^{\beta}\, dx$$
$$= \int_{-1}^{1} [(1-x)^{\alpha+1}(1+x)^{\beta+1} y'\bar{z}' + (1-x)^{\alpha}(1+x)^{\beta} y\bar{z}]\, dx,$$

or, in the notation of inner products, $(\tilde{\mathscr{L}}_S y, z)_{L^2} = \langle y, z \rangle_{H^1}$.

Let y and z be in $\mathscr{D}_{\mathscr{P}_S}$, then the extended Green's second formula for $\alpha < 0$, $\beta < 0$ is given by

$$\int_{-1}^{1} [(\tilde{\mathscr{L}}_S y)\, \bar{z} - y(\tilde{\mathscr{L}}_S \bar{z})](1-x)^{\alpha}(1+x)^{\beta}\, dx = 0,$$

or, $(\tilde{\mathscr{L}}_S y, z)_{L^2} - (y, \tilde{\mathscr{L}}_S z)_{L^2} = 0.$

Let y and z be in $\mathscr{D}_{\mathscr{P}_S}$, then the extended Green's third formula for the parameters $\alpha < 0$, $\beta < 0$ is given by

$$\int_{-1}^{1} (\tilde{\mathscr{L}}_S y)(\tilde{\mathscr{L}}_S \bar{z})(1-x)^{\alpha} (1+x)^{\beta}\, dx$$

$$= \int_{-1}^{1} [(1-x)^{\alpha+1} (1+x)^{\beta+1} y'(\tilde{\mathscr{L}}_S \bar{z})' + (1-x)^{\alpha} (1+x)^{\beta} y(\tilde{\mathscr{L}}_S \bar{z})]\, dx,$$

or, $(\tilde{\mathscr{L}}_S y, \tilde{\mathscr{L}}_S z)_{L^2} = \langle y, \tilde{\mathscr{L}}_S z \rangle_{H^1}$.

Let y and z be in $\mathscr{D}_{\mathscr{P}_S}$, then the extended Green's fourth formula for $\alpha < 0$, $\beta < 0$ is given by

$$\int_{-1}^{1} [(1-x)^{\alpha+1} (1+x)^{\beta+1} (\tilde{\mathscr{L}}_S y)' \bar{z}' + (1-x)^{\alpha} (1+x)^{\beta} (\tilde{\mathscr{L}}_S y) \bar{z}]\, dx$$

$$= \int_{-1}^{1} [(1-x)^{\alpha+1} (1+x)^{\beta+1} y'(\tilde{\mathscr{L}}_S \bar{z})' + (1-x)^{\alpha} (1+x)^{\beta} y(\tilde{\mathscr{L}}_S \bar{z})]\, dx,$$

or, $\langle \tilde{\mathscr{L}}_S y, z \rangle_{H^1} = \langle y, \tilde{\mathscr{L}}_S z \rangle_{H^1}$.

THEOREM 3.17. *$\tilde{\mathscr{L}}_S$ is densely defined in $H^1(-1, 1; (1-x)^{\alpha+1} (1+x)^{\beta+1}, (1-x)^{\alpha} (1+x)^{\beta})$ and is symmetric.*

The spectrum of $\tilde{\mathscr{L}}_S$ is discrete, consisting of $\{n(\alpha+\beta+n+1)+1-\alpha-\beta\}_{n=0}^{\infty}$. The associated eigenfunctions are $\{(1-x)^{-\alpha} (1+x)^{-\beta} P_n^{(-\alpha, -\beta)}(x)\}_{n=0}^{\infty}$, the $\alpha\beta$-shifted Jacobi polynomials.

Hence the normalized eigenfunctions in $H^1(-1, 1; (1-x)^{\alpha+1} (1+x)^{\beta+1}, (1-x)^{\alpha} (1+x)^{\beta})$, when $\alpha < 0$, $\beta < 0$, are

$$\{(1-x)^{-\alpha} (1+x)^{-\beta} P_n^{(-\alpha, -\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha-\beta}\}_{n=0}^{\infty}.$$

THEOREM 3.18. *The resolvent operator $(\tilde{\mathscr{L}}_S - \lambda)^{-1}$ exists if $\lambda \neq n(\alpha+\beta+n+1)+1-\alpha-\beta$, where $n = 0, 1, 2, \ldots$ and is a bounded operator. If λ is real, then $(\tilde{\mathscr{L}}_S - \lambda)^{-1}$ is a bounded, self-adjoint operator. In particular, if $\lambda = 0$, $\tilde{\mathscr{L}}_S^{-1}$ exists as a bounded, self-adjoint operator in $H^1(-1, 1; (1-x)^{\alpha+1} (1+x)^{\beta+1}, (1-x)^{\alpha} (1+x)^{\beta})$.*

THEOREM 3.19. *$\tilde{\mathscr{L}}_S$ is a positive, self-adjoint operator on $\mathscr{D}_{\mathscr{P}_S} \subset H^1$.*

The following spectral resolutions hold:

THEOREM 3.20. *If $I \in H^1(-1, 1; (1-x)^{\alpha+1} (1+x)^{\beta+1}, (1-x)^{\alpha} (1+x)^{\beta})$, and $y \in \mathscr{D}_{\mathscr{P}_S}$, then*

$$
\begin{aligned}
I=\sum_{n=0}^{\infty} & \langle I,(1-x)^{-\alpha}(1+x)^{-\beta} \\
& \times P_n^{(-\alpha,-\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha-\beta}\rangle \\
& \times ((1-x)^{-\alpha}(1+x)^{-\beta} \\
& \times P_n^{(-\alpha,-\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha-\beta}), \\
\tilde{\mathscr{L}}_S\, y=\sum_{n=0}^{\infty} & \{n(\alpha+\beta+n+1)+1-\alpha-\beta\} \\
& \times\langle y,(1-x)^{-\alpha}(1+x)^{-\beta} \\
& \times P_n^{(-\alpha,-\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha-\beta}\rangle \\
& \times ((1-x)^{-\alpha}(1+x)^{-\beta} \\
& \times P_n^{(-\alpha,-\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha-\beta}), \\
(\tilde{\mathscr{L}}_S-\lambda)^{-1} I=\sum_{n=0}^{\infty} & \frac{1}{\{n(\alpha+\beta+n+1)+1-\alpha-\beta\}-\lambda} \\
& \times\langle I,(1-x)^{-\alpha}(1+x)^{-\beta} \\
& \times P_n^{(-\alpha,-\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha-\beta}\rangle \\
& \times ((1-x)^{-\alpha}(1+x)^{-\beta} \\
& \times P_n^{(-\alpha,-\beta)}(x)/\sqrt{n(\alpha+\beta+n+1)+1-\alpha-\beta}),
\end{aligned}
$$

where $\alpha<0$, $\beta<0$.

Not only are the expansions new, the setting in H^1 is also new.

REFERENCES

1. W. N. EVERITT, Legendre polynomials and singular differential operators, *in* "Lecture Notes in Math.," Vol. 827, pp. 83–106, Springer-Verlag, New York/Berlin, 1978.
2. W. N. EVERITT AND L. L. LITTLEJOHN, Differential operators and the Legendre type polynomials, *Differential Integral Equations* **1** (1988), 97–116.
3. W. N. EVERITT, L. L. LITTLEJOHN, AND A. M. KRALL, On some properties of the Legendre type differential expression, *Quaestiones Math.* **13** (1990), 83–106.
4. W. N. EVERITT AND S. D. WRAY, On quadratic integral inequalities associated with second order symmetric differential expressions, *in* "Lecture Notes in Math.," Vol. 1032, pp. 170–223, Springer-Verlag, New York/Berlin, 1982.
5. D. JACKSON, "Fourier Series and Orthogonal Polynomials," Math. Assn. Amer., 1941.
6. H. G. KAPER, M. K. KWONG, AND A. ZETTL, Characterizations of the Friedrichs extensions of singular Sturm–Liouville expressions, *SIAM J. Math. Anal.* **17** (1986), 772–777.

7. A. M. KRALL, $M(\lambda)$ theory for singular Hamiltonian systems with one singular point, *SIAM J. Math. Anal.* **20** (1989), 644–700.
8. A. M. KRALL, Regular left-definite boundary value problems of even order, *Quaestiones Math.* **15** (1992), 105–118.
9. A. PLEIJEL, Generalized Weyl circles, *in* "Lecture Notes in Math.," Vol. 415, pp. 211–226, Springer-Verlag, New York/Berlin, 1974.
10. A. PLEIJEL, A generalization of Weyl's method to pairs of symmetric ordinary differential operators, *in* "Lecture Notes in Math.," Vol. 448, pp. 256–272, Springer-Verlag, New York/Berlin, 1975.
11. G. SZËGO, "Orthogonal Polynomials," Amer. Math. Soc., Providence, RI, 1939.
12. E. C. TITCHMARSH, "Eigenfunction Expansions," Vol. 1, Oxford, 1962.
13. H. WEYL, Über gewöhnliche linear Differentialgleichungen mit singularitären Stellen und Iher Eigenfunktionen, *Akad. Wiss. Gott. Math. Klasse Nachr.* **5** (1910), 442–467.